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On a Characterization of Bilinear Forms Graphs

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We show that the bilinear forms graphs $H_q(n, d)$ of diameter $d \geq 3$ are characterized as distance-regular graphs by their parameters provided that either $n \geq d+3$ and $q \geq 3$, or $n \geq d+4$ and $q = 2$. As a corollary of the method used, we can show the following. If Γ is a distance-regular graph with classical parameters (d, q, α, β) and diameter $d \geq 3$, then either Γ is a Johnson graph, a Grassmann graph, a Hamming graph, or a bilinear forms graph, or β is bounded in terms of d, q and α.

1. INTRODUCTION

We consider only finite, simple and undirected graphs Γ. The *distance* of two vertices v and w is denoted by $d(v, w)$, and the maximum distance that occurs between two vertices of Γ is the *diameter* of Γ, denoted by d. We put $\Gamma_i(v) = \{w \mid d(v, w) = i\}$. The graph Γ is *distance regular* with *intersection array* $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$, if for any two vertices v and w at distance i, we have $c_i = |\Gamma_{i-1}(v) \cap \Gamma_1(w)|$, $b_i = |\Gamma_{i+1}(v) \cap \Gamma_1(w)|$, and $a_i = |\Gamma_i(v) \cap \Gamma_1(w)|$; here $c_0 := b_d := 0$ and $a_i := b_0 - b_i - c_i$. The distance-regular graph Γ has *classical parameters* (d, q, α, β), if it has diameter d, and if

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \qquad \text{and} \qquad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right)$$

for $i = 0, \ldots, d$. Here $\begin{bmatrix} i \\ j \end{bmatrix}$ denotes the Gaussian coefficient with basis q; in particular, we have $\begin{bmatrix} i \\ 1 \end{bmatrix} = \sum_{k=0}^{i-1} q^k$. Throughout, we use the abbreviation $r := \begin{bmatrix} d \\ 1 \end{bmatrix}$.

Many distance-regular graphs related to classical groups and groups of Lie type have classical parameters. A typical example is the *Grassmann graph* $J_q(n, d)$, whose vertices are the subspaces of rank d of a vector space V of rank $n \geq 2d \geq 4$ over the field F_q, where two vertices are adjacent if they meet in a subspace of V of rank $d-1$. It has classical parameters $(d, q, q, \begin{bmatrix} n-d+1 \\ 1 \end{bmatrix} - 1)$. For integers $n \geq d \geq 2$, a vector space of rank $d+n$ and a fixed subspace N of rank n, the *bilinear forms graph* $H_q(n, d)$ is the subgraph of $J_q(n+d, d)$ induced on the vertices that meet N trivially; it has classical parameters $(d, q, q-1, q^n - 1)$. We remark that the Johnson graph $J(n, d)$ has classical parameters $(d, 1, 1, n-d)$ $(n \geq 2d \geq 4)$ and the Hamming graph $H(d, n)$ has classical parameters $(d, 1, 0, n-1)$ $(n, d \geq 2)$.

The Hamming graphs $H(d, n)$ have been characterized (by their parameters) for $d \geq 2$ and $n \neq 4$ by Egawa [5]. The Johnson graphs $J(n, d)$ have been characterized for $(d, n) \neq (2, 8)$ independently by Neumaier [10] and Terwilliger [11]. The Grassmann graphs are characterized for $n \neq 2d, 2d+1$ and $d \geq 3$ (with some exceptions if $q = 2$ or $q = 3$) in [9]. The bilinear forms graphs have been characterized for $n \geq 2d \geq 6$ and $q \geq 4$ by Huang [7] and Cuypers [4]. Thus all these graphs have satisfactory characterizations, except that the bound $n \geq 2d$ for the bilinear forms graphs is quite weak. It is the purpose of this paper to improve this bound.

THEOREM 1.1. *Let Γ be a distance-regular graph with classical parameters (d, q, α, β) where $\alpha = q-1$ and $d \geq 3$. Suppose that either $q = 2$ and $\beta \geq q^{d+4} - 1$ or $q \geq 3$ and $\beta \geq q^{d+3} - 1$. Then q is a prime power, $\beta = q^n$ for some integer n, and Γ is the bilinear forms graph $H_q(n, d)$.*

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COROLLARY 1.2. *The bilinear forms graphs $H_q(n, d)$ of diameter $d \geq 3$ are uniquely determined as distance-regular graphs by their parameters provided that either $q = 2$ and $n \geq d + 4$ or $q \geq 3$ and $n \geq d + 3$.*

The four families of distance-regular graphs mentioned above have the property that β can be arbitrary large with respect to the other three parameters d, q, and α. The following result shows that there are no other graphs with this property.

COROLLARY 1.3. *Suppose that the distance-regular graph Γ has classical parameters (d, q, α, β) with $d \geq 3$ and that Γ is not a Grassmann graph, a bilinear forms graph, a Hamming graph, or a Johnson graph. Then β is bounded in terms of d, q and α. More precisely, if $r = 1 + q + q^2 + \ldots + q^{d-1}$, then we have:*

(a) If $q < 1$, then $|\beta| \leq 1 + |\alpha|(|r| + |q| - 1)|$.
(b) If $q \geq 1$ and $\alpha \neq q, q - 1$, then $|\beta - \alpha + q| \leq r(r - 1)(q + 1)|\alpha - q| \cdot |\alpha + 1 - q|$.
(c) If $q \geq 1$ and $\alpha = q$, then $\beta < 8r(q^2 + 2q)/3$.
(d) If $q \geq 1$ and $\alpha = q - 1$, then $\beta < (2q^4 + 2q^3 + 2q^2 + q - 1)r/(2q - 1)$.

REMARK 1.4. Suppose that Γ is a distance-regular graph of diameter $d = 2$, that is, Γ is a strongly regular graph. It is easy to see that Γ has classical parameters $(2, q, \alpha, \beta)$ and these can be uniquely chosen in such a way that $\beta > 0$. If the parameters are integers and if β is sufficiently large with respect to α and q, then Bose [1] (see also [8]) showed that the graph is the collinearity graph of a partial geometry with parameters $(q + 1, \beta + 1, \alpha + 1)$. This is the analogue to Corollary 1.3 in the case $d = 2$.

2. BILINEAR FORMS GRAPHS

We need the following special case of Corollary 1.3 in [8], which was proved by an improved Bose–Lasker argument. Recall that a *clique* of a graph Γ is a set of mutually adjacent vertices of Γ.

RESULT 2.1. Let $c_2 \geq 1$ and k, λ, s be integers. Suppose that Γ is a regular graph of valency k such that non-adjacent vertices have at most c_2 common neighbours, and adjacent vertices have λ common neighbours. Suppose furthermore that

(a) $\lambda > (2s - 1)(c_2 - 1) - 1$ and
(b) $k < (s + 1)(\lambda + 1) - \frac{1}{2}s(s + 1)(c_2 - 1)$.

Define a line to be a maximal clique C satisfying $|C| \geq \lambda + 2 - (s - 1)(c_2 - 1)$. Then every vertex is in at most s lines, and any two adjacent vertices occur together in a unique line.

For a distance-regular graph with classical parameters (d, q, α, β) with $d \geq 3$, we put

$$r := \begin{bmatrix} d \\ 1 \end{bmatrix} \qquad \text{and} \qquad \lambda := b_0 - a_1 - c_1 = \beta - 1 + (r - 1)\alpha.$$

Notice that every vertex has $b_0 = r\beta$ neighbours and adjacent vertices have λ common neighbours. Proposition 6.2.1 in [2] shows that q is an integer with $q \neq 0, -1$. Hence, if $\alpha = q - 1$, then

$$q \text{ is an integer, } q \neq 0, -1 \text{ and } c_2 = q(q + 1) \geq 2.$$

PROPOSITION 2.2. *Let Γ be a distance-regular graph with classical parameters (d, q, α, β) of diameter $d \geq 3$ with $\alpha = q - 1 \geq 1$. Suppose that there exists an integer $s \geq r$ such that the following properties are fulfilled.*

(a) *If $q = 2$ and $d = 3$, then $s = r = 7$.*
(b) $(s+1)(\lambda+1) - \frac{1}{2}s(s+1)(q^2+q-1) > r\beta$
(c) $\lambda + 1 > s(q^3+q^2+2q-1) - q^2(q^2+q+1)$.

Then q is a prime power, $\beta = q^n - 1$ for some integer $n \geq d$ and Γ is the bilinear forms graph $H_q(n, d)$.

We prove this proposition in several lemmas. By a *line* we shall mean a maximal clique C of Γ satisfying $|C| \geq \lambda + 2 - (s-1)(c_2 - 1)$.

LEMMA 2.3. $s \geq r \geq q^2+q+1$ *and* $\lambda+1 > s(q^3+2q) - q^2 - q - 1$ *and* $\beta > s(q^3+q)$.

PROOF. As $d \geq 3$, we have $s \geq r \geq q^2+q+1$. The bound for $\lambda+1$ follows immediately from hypothesis (C). As $\lambda + 1 - \beta = (r-1)(q-1) \leq s(q-1)$, it follows that $\beta > s(q^3+q) + s - q^2 - q - 1 \geq s(q^3+q)$. □

LEMMA 2.4.

(a) *Every vertex is on at most s lines.*
(b) *Every two adjacent vertices v_1 and v_2 lie in a unique line, denoted by v_1v_2.*
(c) *Every line has at most $\beta + 1$ vertices.*
(d) *Every vertex v is on at least r lines with equality iff all lines on v have $\beta + 1$ vertices.*

PROOF. As $c_2 = q^2 + q$, it follows from hypothesis (b) in Proposition 2.2 and Lemma 2.3 that conditions (a) and (b) of Result 2.1 are satisfied. This proves (a) and (b).

The eigenvalues of Γ are given by $\Theta_i := b_i/q^i - (q^i - 1)/(q-1)$, $i = 0, \ldots, d$, (see Theorem 8.4.2 in [2]). Hence, $\Theta_d = -r$ is the smallest eigenvalue of Γ. It follows from a result of Hoffman [6] (see also Proposition 4.4.6 in [2]) that every clique (and thus every line) of Γ has at most $1 - b_0/\Theta_d = \beta + 1$ vertices. This proves (c). As every vertex has $b_0 = r\beta$ neighbours, part (d) is a consequence of parts (b) and (c). □

The *distance* from a vertex v to a line L is the minimum distance from v to a vertex of L. It will be denoted by $d(v, L)$. For two vertices v and w at distance two, we denote by $[v, w]$ the set consisting of all lines L on v with the property that $d(w, x) \leq 2$ for all $x \in L$.

LEMMA 2.5.

(a) *If $d(v, w) = 2$, then $|[v, w]| \leq q + 1$.*
(b) *If a vertex has distance one from a line, then it has at most $q + 1$ neighbours on that line.*
(c) *If a vertex v lies on t lines, then $|L| \geq \lambda + 2 - (t-1)q$ for every line L on v.*
(d) $|L| \geq \lambda + 2 - (s-1)q$ *for every line L.*

PROOF. By n we denote the maximum integer such that there exist vertices v_1 and v_2 at distance two with $|[v_1, v_2]| = n + 1$. As vertices at distance two have c_2 common neighbours, we have $n \leq c_2 - 1$.

Consider a vertex v_1 and a line L with $d(v_1, L) = 1$. As L is a maximal clique, there exists $v_2 \in L$ with $d(v_1, v_2) = 2$. As the lines on v_1 that meet L are in $[v_1, v_2]$, it follows that

$|\Gamma_1(v_1) \cap L| \le |[v_1, v_2]| \le n+1$. Hence, every vertex that has distance one from a line has at most $n+1$ neighbours on that line.

Consider a vertex v and suppose that v is on t lines. Consider any line L on v and a second vertex $w \in L$. Then every other line on v contains at most n neighbours of w other than v. As v lies on t lines, it follows that v and w have at most $|L| - 2 + (t-1)n$ common neighbours. Hence $|L| \ge \lambda + 2 - (t-1)n$ for every line L on a vertex that lies on t lines. As every vertex is on at most s lines, it follows that every line has at least $\lambda + 2 - (s-1)n$ vertices.

It remains to show $n \le q$. Assume on the contrary that $n \ge q+1$. Consider vertices v and w at distance two such that $|[v, w]| = n+1$, and denote by M the set consisting of the neighbours of v that have distance one or two from w. Then $|M| = b_0 - b_2 = (q+1)\lambda - (q+1)(q^2-q-1)$.

As every line has at least $\lambda + 2 - (s-1)n$ vertices, every line of $[v, w]$ has at least $\lambda + 1 - (s-1)n$ vertices in M. If we define $f(x) := (x+1)(\lambda + 1 - (s-1)x)$, then it follows that the $n+1$ lines of $[v, w]$ cover at least $f(n)$ vertices of M. Hence $f(n) \le |M|$. As $q+1 \le n \le c_2 - 1$, we have $f(n) \ge \min\{f(q+1), f(c_2-1)\}$. As $c_2 = q^2 + q$, we have $f(c_2-1) - f(q+1) = (q^2-2)[\lambda + 1 - (q+1)^2(s-1)] \ge 0$. Hence $f(q+1) \le f(n)$ and thus

$$(q+2)(\lambda + 1 - (s-1)(q+1)) = f(q+1) \le |M| = (q+1)(\lambda+1) - (q+1)(q^2-q-1).$$

This is equivalent to

$$\lambda + 1 \le s(q+1)(q+2) - (q+1)(q^2+1).$$

Using Lemma 2.3, it follows that $q^3 + 2q < (q+1)(q+2)$. As $q \ge 2$, this is a contradiction. This proves that $n \le q$. □

LEMMA 2.6. *Every vertex is on at most* $9r/8$ *lines.*

PROOF. Suppose that v is on t lines. Since v has $r\beta$ neighbours, it follows from Lemma 2.5 (c) that

$$t[\lambda + 1 - (t-1)q] \le r\beta.$$

As $\lambda + 1 = \beta + (r-1)(q-1)$, this is equivalent to

$$(t-r)[\beta - t(q-1)] \le t(t-1).$$

As $t \le s$, we obtain $\beta > t(q^3+q)$ from Lemma 2.3. It follows that $(t-r)t(q^3+1) \le t(t-1)$. As $q \ge 2$, this implies that $9(t-r) \le t$, or $8t \le 9r$. □

LEMMA 2.7. *If* $d(v_1, v_2) = 3$, *then* $|\Gamma_2(v_1) \cap L| \le q(q^2+q+1)$ *for every line* L *on* v_2.

PROOF. Put $X = \Gamma_2(v_1) \cap L$ and $W = \Gamma_1(v_1) \cap \Gamma_2(v_2)$. If $x \in X$, then $\Gamma_1(x) \cap \Gamma_1(v_1) \subseteq W$. Hence, every vertex of X has c_2 neighbours in W. Counting pairs of adjacent vertices $x \in X$ and $w \in W$ gives

$$|X|c_2 = \sum_{w \in W} |\Gamma_1(w) \cap L|.$$

Using Lemma 2.5 (b), we obtain $|X|c_2 \le |W|(q+1)$. As $|W| = c_3 = q^2(q^2+q+1)$ and $c_2 = q^2 + q$, the assertion follows. □

LEMMA 2.8. *If* $d(v, w) = 2$, *then* $|[v, w]| = q+1$.

PROOF. Denote by M the set consisting of the neighbours of v that have distance one or two from w. Then $|M| = b_0 - b_2 = (q+1)(\lambda+1) - (q+1)(q^2 - q - 1)$. Put $c := q(q^2+q+1)$. Lemma 2.5 shows that $|[v, w]| \leq q + 1$.

Assume that $|[v, w]| \leq q$. By Lemma 2.4, every line on v contains at most $\beta + 1$ vertices and thus at most β vertices in M. A line on v that is not in $[v, w]$ contains a vertex at distance three from w and contains therefore, by Lemma 2.7, at most $c - 1$ vertices in M. As v is on at most s lines, it follows that

$$|M| \leq q\beta + (s - q)(c - 1) = q(\lambda + 1 - (r - 1)(q - 1)) + (s - q)(c - 1).$$

As $|M| = (q + 1)(\lambda + 1) - (q + 1)(q^2 - q - 1)$, this gives

$$\lambda + 1 \leq (q + 1)(q^2 - q - 1) - (r - 1)q(q - 1) + (s - q)(c - 1).$$

As $d \geq 3$, we have $r - 1 \geq q^2 + q \geq q + 1$. Hence $\lambda + 1 \leq (s - q)(c - 1)$. This contradicts hypothesis (c) in Proposition 2.2. □

LEMMA 2.9. *Suppose that every point is on exactly r lines. Then $|\Gamma_1(v) \cap L| = q$ for every vertex v and every line L with $d(v, L) = 1$.*

PROOF. This can be shown using the argument of the proof of Lemma 2.3 in [7]. We give a slightly different proof.

As every vertex is on r lines, Lemma 2.4 shows that every line has $\beta + 1$ vertices. Fix a vertex v and denote by T the set consisting of all triples (u, u', w) of distinct pairwise collinear vertices with $d(v, u) = d(v, u') = 1$ and $d(v, w) = 2$. Furthermore, put $k_i = |\Gamma_i(v)|$ for $i = 1, 2$.

If $d(v, w) = 2$, then v and w have $q^2 + q$ common neighbours, which are on the $q + 1$ lines of $[w, v]$. It follows that w occurs in at least $(q + 1)q(q - 1)$ triples of T with equality iff every line of $[w, v]$ contains exactly q neighbours of x. Hence $|T| \geq k_2 q(q^2 - 1)$.

Now consider a neighbour u of v. Then u and v have $\lambda - (\beta - 1) = (r - 1)(q - 1)$ common neighbours that do not lie on the line uv. These lie on one of the $r - 1$ lines $L_2, \ldots, L_r$ other than uv on u. If $|\Gamma_1(v) \cap L_i| = 1 + x_i$, then $\sum x_i = (r - 1)(q - 1)$ and u occurs in $\sum x_i(\beta - x_i)$ triples of T. As $\sum x_i^2 \geq (\sum x_i)^2/(r - 1) = (r - 1)(q - 1)^2$, it follows that u occurs in at most $\beta(r - 1)(q - 1) - (r - 1)(q - 1)^2$ triples of T with equality iff all lines other than uv on u contain exactly q neighbours of v. Hence $|T| \leq k_1(r - 1)(q - 1)(\beta + 1 - q)$.

Comparing both bounds for T using $k_2 = k_1 b_1/c_2$, we obtain equality. Hence, every line at distance one from v contains exactly q neighbours of v. □

LEMMA 2.10. *Consider a vertex p and a line L with $d(p, L) = 1$. Then $[p, v_1] = [p, v_2]$ for all vertices $v_1, v_2 \in L \cap \Gamma_2(p)$.*

PROOF. Assume that there exist vertices $v_1, v_2 \in L \cap \Gamma_2(p)$ with $[p, v_1] \neq [p, v_2]$. Put $L_i := \Gamma_i(p) \cap L$ and $n_i := |L_i|$, $i = 1, 2$, and $T := [p, v_1] \cap [p, v_2]$. Then $n_1 \leq t := |T| \leq q$. Denote by T_2 the set consisting of the lines on p that are not in T. Then $|T_2| \leq s - t$, since p lies on at most s lines.

Consider a line $H \in T_2$. W.l.o.g. $H \notin [p, v_1]$. Then there exists $h \in H$ with $d(v_1, h) = 3$. Lemma 2.7 shows that $|\Gamma_2(h) \cap L| \leq c := q(q^2 + q + 1)$. As $L_1 \subseteq \Gamma_2(h) \cap L$, we obtain $|\Gamma_2(h) \cap L_2| \leq c - n_1$. It follows that there exist at most $c - n_1$ vertices $v \in L_2$ with $H \in [p, v]$. On the other hand, if $v \in L_2$, then $[p, v]$ contains at least $q + 1 - t$ lines of T_2. Counting pairs (v, H) with $v \in L_2$ and $H \in T_2 \cap [p, v]$, we obtain

$$n_2(q + 1 - t) \leq |T_2|(c - n_1) \leq (s - t)(c - n_1).$$

As $t \leq q$ and $s \geq r > q+1$, we have $s - t \leq (q+1-t)(s-q)$. It follows that $n_2 \leq (s-q)(c-n_1)$. As $d(p, L) = 1$, we have $n_1 \geq 1$. Hence $|L| = n_1 + n_2 \leq 1 + (s-q)(c-1)$. From Lemma 2.5 we obtain $|L| \geq \lambda + 2 - (s-1)q$. Hence

$$\lambda + 1 \leq (s-q)(c-1) + (s-1)q = s(c+q-1) - qc.$$

As $c = q(q^2+q+1)$, this contradicts hypothesis (c) in Proposition 2.2. $\square$

Next we verify the dual of Pasch's Axiom. This means that for all adjacent points v and v' we have: if X is the set consisting of all vertices that are not on the line vv' and which are adjacent to v and v', then X is a clique.

LEMMA 2.11. *The incidence structure consisting of the vertices and lines satisfies the dual of Pasch's Axiom.*

PROOF. Consider two adjacent vertices v and v', put $H = vv'$, and denote by X the set consisting of all common neighbours of v and v' that are not on H. We have to show that X is a clique. As $|H| \leq \beta + 1$ (Lemma 2.4) and $|H| - 2 + |X| = \lambda$, we have $|X| \geq \lambda + 1 - \beta = (r-1)(q-1) = q^d - q$.

Consider $x \in X$. We shall first show that $|\Gamma_1(x) \cap X| \geq |X| - 1 - q(q-1)$. Put $L = vx$. Then there exists a vertex $p \in L$ with $d(p, v') = 2$. Consider any vertex $x' \in X$ with $x' \neq x$ and put $L' := v'x'$. Then $L' \neq H$. If $L' = v'x$, then clearly $x' \sim x$. If x' is on one of the $q-1$ lines of $[v', p]$ other than H and $v'x$, then x' might not be adjacent to x. As each of these $q-1$ lines contains at most q common neighbours of v and v' (Lemma 2.5), this gives rise to at most $(q-1)q$ vertices in X that are not adjacent to x. Finally, consider the case that $L' \notin [v', p]$. Then $d(x', p) \geq 2$. As $d(x', L) = 1$, it follows that $d(x', p) = 2$. This implies that $x \sim x'$ (otherwise $d(x', x) = 2$ and Lemma 2.10 would give $[x', p] = [x', x]$; however $L' \in [x', x]$, as $d(x, L') = 1$, but $L' \notin [x', p]$, as $L' \notin [v', p]$). This proves that $|\Gamma_1(x) \cap X| \geq |X| - 1 - q(q-1)$.

Now consider two distinct vertices $x_1, x_2 \in X$, put $X_i := X \cap \Gamma_1(x_i)$, $i = 1, 2$, and assume that $x_1 \not\sim x_2$. Then $|X_i| \geq |L| - q(q-1) - 1$ and $x_1, x_2 \notin X_1 \cup X_2$. Hence

$$\begin{aligned} |X_1 \cap X_2| &\geq |X_1| + |X_2| - |X_1 \cup X_2| \geq 2(|X| - q(q-1) - 1) - (|X| - 2) \\ &= |X| - 2q(q-1). \end{aligned}$$

As $v, v' \in \Gamma_1(x_1) \cap \Gamma_1(x_2)$, it follows that $|\Gamma_1(x_1) \cap \Gamma_1(x_2)| \geq |X| + 2 - 2q(q-1)$. On the other hand x_1 and x_2 have at most c_2 common neighbours, because $d(x_1, x_2) = 2$. Hence $|X| \leq c_2 + 2q(q-1) - 2 = 3q^2 - q - 2$. From $|X \geq q^d - q$, we obtain $q^d \leq 3q^2 - 2$. As $d \geq 3$ and $q \geq 2$, it follows that $d = 3$ and $q = 2$.

Hypothesis (a) in Proposition 2.2 says that $s = r = 7$ in this case, that is, every vertex lies on precisely r lines. Lemma 2.9 shows that a vertex that has distance one from a line has precisely $q = 2$ neighbours on this line. For $x \in X$, we can now improve the above bound to $|\Gamma_1(x) \cap X| \geq |X| - 1 - (q-1)^2 = |X| - 2$. As x_1 and x_2 are not adjacent, it follows that x_1 and x_2 are adjacent to every vertex of $X \setminus \{x_1, x_2\}$. Consider the line $L_1 := vx_1$. Lemma 2.9 shows that x_1 and v are the only neighbours of v' on L, and that x_2 has two neighbours on L. Hence, there exists a vertex $w \neq v$ on L that is adjacent to x_2 but not to v'. But now the vertices v, v', w and the vertices of $X \setminus \{x_1, x_2\}$ are adjacent to x_1 and x_2. Since $c_2 = (q+1)q = 6$ and $|X| \geq q^d - q = 6$, this is a contradiction. $\square$

Every maximal clique that is not a line will be called an *assembly*.

LEMMA 2.12.

(a) Two adjacent vertices v and v' lie in a unique assembly. It contains all common neighbours of v and v' that are not on the line vv'.
(b) If a line L and an assembly B share two vertices, then $|L| + |B| - |L \cap B| = \lambda + 2$.

PROOF. (a) Denote by X the set consisting of all common neighbours of v and v' that are not on $L := vv'$. As $\lambda = \beta - 1 + (r-1)(q-1) > \beta - 1 \geq |L| - 2$, we have $X \neq \emptyset$.

By the dual of Pasch's Axiom, X is a clique. The dual of Pasch's Axiom also shows that every vertex $l \in L$ that is adjacent to some vertex x of X is also adjacent to every other vertex y of X, because l and y do not lie on one of the lines xv and xv'. This shows that $X \cup \{v, v'\}$ lies in a unique maximal clique B, which consists of X and all vertices of L that are adjacent to one (and thus all) vertices of X.

This implies that every clique that contains v, v' and some vertex of X is contained in B. Hence, B is the only assembly containing v and v'.

(b) Part (b) follows from (a). □

LEMMA 2.13. *Suppose that the lines L_1 and L_2 meet in a vertex v. Then either every vertex of L_1 has a neighbour on $L_2 \setminus \{v\}$, or every vertex of L_2 has a neighbour on $L_1 \setminus \{v\}$.*

PROOF. Assume on the contrary that there exist vertices $v_i \in L_i$ that have no neighbour on $L_{3-i} \setminus \{v\}$, $i = 1, 2$. Then v_1 and v_2 have distance two and thus $c_2 = q^2 + q$ common neighbours. Each common neighbour is the meet of a line of $[v_1, v_2]$ and a line of $[v_2, v_1]$. As $L_1 \in [v_1, v_2]$, $L_2 \in [v_2, v_1]$, and $|[v_1, v_2]| = |[v_2, v_1]| = q + 1$, it follows that v_1 and v_2 have at most $q^2 + 1$ common neighbours. But $c_2 = q^2 + q > q^2 + 1$, a contradiction. □

LEMMA 2.14.

(a) A line and an assembly meet in at most $q + 1$ vertices.
(b) If two lines meet, then their cardinalities differ by at most $q - 1$.
(c) Every vertex lies on the same number of lines.

PROOF. (a) Consider an assembly B that meets the line L. Then B contains a vertex v with $v \notin L$. As v has at most $q + 1$ neighbours on L, we have $|B \cap L| \leq q + 1$.

(b) Consider two distinct lines L_1 and L_2 containing the vertex v. In view of Lemma 2.13, we can find adjacent vertices $v_1 \in L_1$ and $v_2 \in L_2$ with $v_1, v_2 \neq v$. Let B be the unique assembly with $v_1, v_2 \in B$. Lemma 2.12 shows that $v \in B$ and that $|L_1| - |L_1 \cap B| = |L_2| - |L_2 \cap B|$. As $2 \leq |L_i \cap B| \leq q + 1$, it follows that $|L_1|$ and $|L_2|$ differ by at most $q - 1$.

(c) It suffices to prove this for adjacent vertices v_1 and v_2. Let s_i be the number of lines of v_i, put $L := v_1 v_2$, and denote by l_{ij}, $j = 1, \ldots, s_i$, the cardinalities of the lines $\neq L$ on v_i. As v_1 and v_2 have the same number of neighbours, we have

$$\sum_{j=1}^{s_1} (l_{1j} - 1) = \sum_{j=1}^{s_2} (l_{2j} - 1).$$

Assume that $s_2 > s_1$. As every line has at least $\lambda + 2 - (s-1)q$ vertices, it follows that

$$\sum_{j=1}^{s_1} (l_{1j} - l_{2j}) = \sum_{j=s_1+1}^{s_2} (l_{2j} - 1) \geq (s_2 - s_1)(\lambda + 2 - (s-1)q).$$

Part (b) shows that $|l_{1j}| \leq |L| + (q-1)$ and $|l_{2j}| \geq |L| - (q-1)$. Hence

$$2s_1(q-1) \geq (s_2 - s_1)(\lambda + 2 - (s-1)q).$$

As $s_1 \leq s_2 - 1 \leq s - 1$, it follows that

$$2(s-1)(q-1) \geq \lambda + 2 - (s-1)q.$$

Consequently $\lambda + 1 < (3q-2)(s-1)$. This contradicts Lemma 2.3. Hence $s_2 \leq s_1$. The same argument shows that $s_1 \leq s_2$. □

LEMMA 2.15. *Consider two lines L_1 and L_2 on a vertex v. If $|L_1| \geq |L_2|$, then every vertex on $L_1 \setminus \{v\}$ has a neighbour other than v on L_2.*

PROOF. For $i = 1, 2$, denote by X_i the set consisting of the points of $L_i \setminus \{v\}$ that have a neighbour on $L_{3-i} \setminus \{v\}$. Let $B_1, \ldots, B_b$ be the assemblies on v that meet both L_1 and L_2 in a second vertex. Then X_i consists of those points $\neq v$ of L_i that lie in some of the assemblies B_j. Thus

$$|X_i| = \sum_{j=1}^{b} (|L_i \cap B_j| - 1), \qquad i = 1, 2.$$

Put $c := |L_1| - |L_2|$. Lemma 2.12 implies that $c = |L_1 \cap B_j| - |L_2 \cap B_j|$ for every assembly B_j. Hence $|X_1| = |X_2| + bc$.

We have to show that $X_1 = L_1 \setminus \{v\}$, that is $|X_1| = |L_1| - 1$. Assume this does not hold. Then there exists a vertex $v_1 \in L_1$ that has no neighbour $\neq v$ on L_2. Lemma 2.13 implies that every vertex of L_2 has a neighbour $\neq v$ on L_1, that is, $|X_2| = |L_2| - 1$. It follows that

$$|L_1| - 1 > |X_1| = |X_2| + bc = |L_2| - 1 + bc.$$

As $b, c \geq 0$, we obtain $|L_1| > |L_2|$, which in turn implies that $c = |L_1| - |L_2| \geq 1$ and $|L_1| \geq |L_2| + b$. Using Lemma 2.14 (b), we obtain $b \leq q - 1$. As, by Lemma 2.14 (a), an assembly and a line share at most $q + 1$ vertices, it follows that

$$|L_2| - 1 = |X_2| = \sum_{j=1}^{b} (|L_2 \cap B_j| - 1) \leq bq \leq q(q-1).$$

As $|L_2| \geq \lambda + 2 - (s-1)q$ (Lemma 2.5), it follows that $\lambda + 1 \leq (s-1)q + q(q-1)$. This contradicts Lemma 2.3. □

LEMMA 2.16. *Consider a vertex v, two lines L and L' and an assembly B with $v \in L, L', B$.*

(a) If $|L \cap B| \geq 2$ and $|L| \geq |L'|$, then $|L' \cap B| = |L \cap B| + |L'| - |L|$.
(b) If $|L \cap B| = q + 1$ and $|L'| > |L|$, then $L' \cap B = \{v\}$.

PROOF. (a) Consider a vertex $w \neq v$ of $B \cap L$. Then B contains all common neighbours of v and w that do not lie on L. As, by Lemma 2.15, w has a neighbour $\neq v$ on L', it follows that $|L' \cap B| \geq 2$. Lemma 2.12 shows that $|L| - |L \cap B| = \lambda + 2 - |B| = |L'| - |L' \cap B|$.

(b) Assume that $|L' \cap B| \geq 2$. Part (a) shows that $|L' \cap B| = |L \cap B| + |L'| - |L|$. As $|L'| > |L|$ and $|L \cap B| = q+1$, we obtain $|L' \cap B| > q+1$. This contradicts Lemma 2.14 (a). □

LEMMA 2.17. *Suppose that the vertex v is on $t > r$ lines. Then every line on v meets some assembly on v in $q + 1$ vertices, and every assembly on v has at least $tq - r + 2$ vertices.*

PROOF. Denote by $l+1$ the minimum length of the lines on v. Lemma 2.14 (b) shows that every line on v has at most at most $l+q$ vertices. As the t lines on v cover the $r\beta$ neighbours of v, it follows that

$$tl \le r\beta \le t(l+q-1).$$

Put $c := \beta - l$. As $r+1 \le t \le 9r/8$ (Lemma 2.6), it follows that

$$\frac{\beta}{r+1} \le \frac{(t-r)\beta}{t} \le c \le \frac{(t-r)\beta}{t} + q - 1 \le \frac{1}{9}\beta + (q-1).$$

As $s \ge t \ge r+1$ (Lemma 2.4 (a)) and $\beta \ge q^2 t$ (Lemma 2.3), we also have

$$c \ge \frac{(t-r)\beta}{t} \ge (t-r)q^2 \ge (t-r)q + q^2 - q > (t-r)q + 2q - 3.$$

Denote by $\mathcal{B}$ the set consisting of all assemblies on v, and consider $B \in \mathcal{B}$. Then B meets some line L on v in at least two vertices. As $|L| \le l+q = \beta + q - c$, Lemma 2.12 shows that

$$\begin{aligned}|B| &= \lambda + 2 + |L \cap B| - |L| \ge \lambda + 4 - |L|\\ &= \beta + 3 + (r-1)(q-1) - |L| \ge (r-2)(q-1) + 2 + c =: b.\end{aligned}$$

As the assemblies of $\mathcal{B}$ cover each of the $r\beta$ neighbours of v exactly once, it follows that $|\mathcal{B}|(b-1) \le r\beta$. As $c > (t-r)q + 2q - 3$, we also have $b > tq - r + 1$, so every assembly on v has at least $tq - r + 2$ vertices.

Now consider any line L on v. Lemma 2.14 (a) says that $|L \cap B| \le q+1$ for every $B \in \mathcal{B}$. Assume that $|L \cap B| \le q$ for all $B \in \mathcal{B}$. As every vertex of $L \setminus \{v\}$ occurs in an assembly of $\mathcal{B}$, it follows that $|\mathcal{B}|(q-1) \ge |L| - 1 \ge l$. Hence

$$l(b-1) \le |\mathcal{B}|(b-1)(q-1) \le r\beta(q-1).$$

As $l = \beta - c$, it follows that

$$(\beta - c)[(r-2)(q-1) + c] \le r\beta(q-1),$$

which can be written in the form

$$c[\beta - (r-2)(q-1) - c] \le 2\beta(q-1).$$

Using the upper and lower bound for c, we obtain

$$\frac{\beta}{r+1}\left[\frac{8}{9}\beta - (r-1)(q-1)\right] \le 2\beta(q-1).$$

Hence $\frac{8}{9}\beta \le (3r+1)(q-1)$. As $r+1 \le t \le s$, we have $\frac{8}{9}\beta \le 3s(q-1)$. This contradicts Lemma 2.3. □

LEMMA 2.18. *Suppose that the vertex v is on $t > r$ lines. Then not all lines on v have the same length.*

PROOF. Assume that every line on v has the same length $l+1$. Put $l = \beta - c$. As v has $r\beta$ neighbours, we have $t(\beta - c) = tl = r\beta$. It follows that $c = (t-r)\beta/t$.

In view of Lemma 2.17, there exists a line L_0 and an assembly B_0 on v with $|L_0 \cap B_0| = q+1$. Part (a) of Lemma 2.16 shows that B_0 meets all lines on v in $q+1$ vertices. Hence $|B_0| = 1 + tq$.

Now consider any assembly B on v. Lemma 2.16 shows that B meets all lines on v in the same number of vertices. Denote this number by $q+1-w$. Then $|B| = 1+t(q+1-w) = |B_0|-tw$. As B meets L_0 in $q+1-w$ vertices and as B_0 meets L_0 in $q+1$ vertices, Lemma 2.12 shows that $|B_0| = |B| + w$. As $|B| = |B_0| - tw$, it follows that $(t-1)w = 0$. Hence $w = 0$.

We have shown that every assembly on v has size $|B_0| = 1 + tq$ and meets all lines on v in $q + 1$ vertices.

Consider any neighbour v' of v and denote by B' the assembly on v and v'. Then every line on v' meets B' in at most $q + 1$ vertices. As $|B'| = 1 + tq$ and as v' is on t lines (Lemma 2.14 (c)), it follows that all lines on v' meet B' in $q + 1$ vertices. Lemma 2.12 (b) implies that all lines on v' have the same length. Hence all lines on v' have length $l+1$. Using an inductive argument, we can show that all lines on a vertex at distance i from v have size $l + 1$. Hence all lines of Γ have size $l + 1$.

The argument used for the assemblies on v shows that all assemblies have the same size $1+qt$ and that a line and an assembly either miss or meet in $q+1$ vertices. Thus, if a vertex w has distance one from a line L, then w has precisely $q+1$ neighbours on L (if $w' \in \Gamma_1(w)\cap L$, and if B' is the assembly on w and w', then $\Gamma_1(w) \cap L = B' \cap L$).

Now consider vertices v_1 and v_2 at distance two. Let u be a common neighbour of v_1 and v_2, and put $L = v_2u$. As u is a neighbour of v_1 on L, we see that v_1 has $q + 1$ neighbours on L. Hence v_1 lies on $q + 1$ lines that contain a vertex on L. As v_2 has a neighbour on each of these lines, it has $q+1$ neighbours on each of these lines. Hence $|\Gamma(v_1)\cap\Gamma_1(v_2)| \geq (q+1)^2$, a contradiction. □

LEMMA 2.19. *Every vertex is on r lines.*

PROOF. By Lemma 2.14, all vertices lie on the same number t of lines. Assume that $t > r$. Consider a vertex v. Denote by $l + 1$ the minimum and by $l + 1 + u$ the maximum size of the lines on v, and denote by m the number of lines of minimum size $l + 1$ on v. In view of the preceding lemma, we have $u \geq 1$. We prove the lemma in five steps.

Step 1. $qm \geq tq - r + 1$.

Consider a line L of size $l+1$ on v. In view of Lemma 2.17, there exists an assembly B on v that meets L in $q+1$ vertices. If H is any line on v, then Lemma 2.16 shows that $B\cap H = \{v\}$ if $|H| > |L|$, and $|B \cap H| = q + 1$, if $|H| = |L|$. As v lies on m lines of size $l + 1$, it follows that $|B| = 1 + mq$. Lemma 2.17 shows that $|B| \geq tq - r + 2$. This proves Step 1.

Step 2. *Every line v has size $l+1$ or $l+2$. If L is a line of size $l+2$ on v, then $L \cap B = \{v\}$ or $|L \cap B| = q + 1$ for every assembly B on v.*

Consider any line L_0 of size $l + 1 + u$ on v and any assembly B on v with $|B \cap L_0| \geq 2$. Put $|B \cap L_0| = q + 1 - w$. If L is any line on v, then Lemma 2.16 shows that L meets B in $q+1-w+|L|-|L_0|$ vertices. Thus B meets the m lines of size $l+1$ on v in $q+1-w-u$ and every other line on v in at most $q + 1 - w$ vertices. As v lies on t lines, it follows that

$$|B| - 1 = \sum_{L\in\mathcal{L}}(|B \cap L| - 1) \leq (t - m)(q - w) + m(q - w - u) = t(q - w) - mu.$$

We have $u \geq 1$ and $w \geq 0$. Assume that $u \geq 2$ or $w \geq 1$. As $t \geq m$, it follows that $|B| - 1 \leq tq - 2m$. Using Lemma 2.17 and Step 1, we obtain

$$q(tq - r + 1) \leq q(|B| - 1) \leq tq^2 - 2mq \leq tq^2 - 2(tq - r + 1).$$

This is equivalent to $2qt \leq (r - 1)(q + 2)$. As $t > r$ and $q \geq 2$, this is a contradiction. Hence $u = 1$, that is every line on v has at most $l + 2$ vertices. Furthermore, $w = 0$, that is $|B \cap L| = q + 1$.

From now on, we call the lines of size $l+2$ *long*, and the lines of size $l+1$ *short*.

Step 3. *Consider an assembly B on v. If B meets all long lines on v only in v, then $|B| = 1+qm$ and B meets every short line on v in $q+1$ vertices. Otherwise $|B| = 1+tq-m$, B meets the long lines on v in $q+1$ and the short lines on v in q vertices.*

First consider the case that B meets all long lines on v only in v. Lemma 2.16 shows that B meets all short lines on v in the same number of vertices. Denote this number by $q+1-w$ with $w \geq 0$. Then $|B| = 1+m(q-w)$. Lemma 2.17 implies that $tq+1-r \leq m(q-w)$. As $t > r$ and $t \geq m$, it follows that $w < 1$, that is, $w = 0$. Hence $|B| = 1+mq$, and B meets all short lines on v in $q+1$ vertices.

Now consider the case that B meets some long line L on v not only in v. Step 2 shows that $|B \cap L| = q+1$. Lemma 2.16 shows that B meets every line H on v in $q+1+|H|-|L|$ vertices, that is, B meets the long lines on v in $q+1$ and the short lines on v in q vertices. Hence $|B| = 1+(t-m)q+m(q-1)$.

Step 4. $(q+1)m = tq+1$ *and* $r\beta = t(l+1)-m$.

Consider a short line L on v. By Lemma 2.17, there exists an assembly B_1 on v that meets L in $q+1$ points. The same lemma shows that there exists an assembly B_2 on v that meets some long line on v in $q+1$ points. Using Step 3, this gives

$$|B_1 \cap L| = q+1, \qquad |B_2 \cap L| = q, \qquad |B_1| = 1+qm, \qquad \text{and} \qquad |B_2| = 1+tq-m.$$

Lemma 2.12 shows that $|B_i|+|L|-|B_i \cap L| = \lambda+2$. Hence $|B_1| = |B_2|+1$. It follows that $2+tq-m = 1+qm$, which gives $(q+1)m = tq+1$. As the $t-m$ long and the m short lines on v cover every neighbour of v once, we have $r\beta = b_0 = (t-m)(l+1)+ml = t(l+1)-m$. This proves Step 4.

Lemma 2.14 (c) says that every vertex lies on t lines. Hence, everything we have proved for v holds for all vertices. Furthermore, Step 4 show that the integers m and l do not depend on v. It follows that every line has size $l+1$ or $l+2$, and that every vertex lies on m short lines.

Step 5. *Suppose that the assembly B meets some long line in more than one vertex, and consider any line X with $B \cap X \neq \emptyset$. Then $|B \cap X| = q+1$ if X is long, and $|B \cap X| = q$ if X is short.*

This is true, because Step 3 holds for every vertex.

Now consider a vertex v and two long lines L_1 and L_2 on v. Choose vertices $v_i \in L_i \setminus \{v\}$ such that $d(v_1, v_2) = 2$. Step 5 shows that the assembly on v_1 and v meets L_2 in $q+1$ vertices. Hence v_1 has $q+1$ neighbours $w_1, \ldots, w_{q+1}$ on L_2. Denote the line on v_1 and w_i by H_i. W.l.o.g. $w_1 = v$, that is $H_1 = L_1$. Step 5 shows that the assembly on v_2 and w_i meets H_i in at least q vertices. Furthermore, as $H_1 = L_1$ is long, the assembly on v_2 and w_1 meets H_1 in $q+1$ vertices. It follows that v_2 has at least q^2+q+1 neighbours on one of the lines $H_1, \ldots, H_{q+1}$. But $d(v_1, v_2) = 2$, so v_1 and v_2 have only $c_2 = q^2+q$ common neighbours, a contradiction.

We have shown that every vertex is on precisely r lines. Lemma 2.4 implies that every line has $\beta+1$ vertices, and Lemma 2.9 shows that if a vertex has distance one from a line, then it has q neighbours on that line. As $c_2 = (q+1)q$, it follows that if two vertices v and w have distance two, then each of the $q+1$ lines of $[w, v]$ contains q neighbours of v.

Now the proof can be completed using the techniques of Huang [7]. In his paper, he proves a characterization of bilinear forms graphs under certain conditions. Notice that he needs the restriction $q \geq 4$ in section 3 of [7] to apply a well-known Theorem of Buekenhout [3]. However, it was noticed by Cuypers [4] that the use of Buekenhouts result can be avoided, if one uses a result of Thas and DeClerck [12], which also holds for $q = 2$ and $q = 3$. Huang also uses the so-called 4-vertex condition. However, he needs it only to show that lines have size $\beta + 1$ and that the incidence structure consisting of the vertices and the lines satisfies the dual of Pasch's Axiom, which we have already shown. Thus the arguments of Huang show that q is a prime power, $\beta = q^n - 1$ for some integer $n \geq d$ and Γ is the bilinear forms graph $H_q(n, d)$. This completes the proof of Proposition 2.2. □

3. GRAPHS WITH CLASSICAL PARAMETERS

Proposition 2.2 requires an integer s satisfying certain conditions. The following lemma gives an existence criterion.

LEMMA 3.1. *There exists an integer s satisfying the hypotheses of Proposition 2.2, provided that one of the following conditions is satisfied.*

(a) $\beta \geq (2q^4 + 2q^3 + 2q^2 + q - 1)r/(2q - 1)$.
(b) $q \geq 3$ *and* $\beta \geq q^{d+3} - 1$.
(c) $q = 2$ *and* $\beta \geq q^{d+4} - 1$.

PROOF. As $\lambda + 1 = \beta + (r - 1)(q - 1)$, conditions (b) and (c) in Proposition 2.2 can be written as

$$\beta > \frac{s+1}{2(s+1-r)}[s(q^2 + q - 1) - 2(r-1)(q-1)], \text{ and} \tag{1}$$

$$\beta > s(q^3 + q^2 + 2q - 1) - q^2(q^2 + q + 1) - (r-1)(q-1). \tag{2}$$

We shall define a real number $s_0 > r$ and then the integer s in such a way that $s_0 - 1 < s \leq s_0$. Then $(s+1)/(s+1-r) \leq s_0/(s_0 - r)$. Hence, instead of inequalities (1) and (2), it suffices to show that

$$\beta > \frac{s_0}{2(s_0 - r)}[s_0(q^2 + q - 1) - 2(r-1)(q-1)], \text{ and} \tag{3}$$

$$\beta > s_0(q^3 + q^2 + 2q - 1) - q^2(q^2 + q + 1) - (r-1)(q-1). \tag{4}$$

We choose $s_0 := 2qr/(2q - 1)$. Then $s_0/(s_0 - r) = 2q$. Hence, (3) and (4) are satisfied provided that

$$\beta \geq s_0(q^3 + q^2 + 2q - 1) - r(q-1) = \frac{(2q^4 + 2q^3 + 2q^2 + q - 1)r}{2q - 1}.$$

This proves part (a). Part (b) is an immediate consequence of part (a). In order to prove part (c), we have to choose s_0 much more carefully. First notice that in the case $q = 2$ and $d = 3$, we can choose $s = r = 7$; then (1) and (2) and hence hypotheses (a), (b) and (c) of Proposition 2.2 are satisfied. Suppose from now on that $q = 2$ and $d \geq 4$. Let u be the larger root of the equation

$$x(5x - 2) = 2(x-1)(15x - 1).$$

Then $u \approx 1.129$ and

$$c := \frac{x(5x-2)}{2(x-1)} = (15x - 1) \approx 15.94.$$

We choose $s_0 := ur$. As $q = 2$, inequalities (3) and (4) require that

$$\beta > \frac{u}{2(u-1)}(5ur - 2r + 2) = cr + \frac{u}{u-1}$$

and

$$\beta > 15ur - 27 - r = cr - 27.$$

If $\beta \geq q^{d+4} - 1$, then $\beta > 16(q^d - 1) + 15 = 16r + 15$ and (3) and (4) are satisfied. This proves part (c). $\square$

Theorem 1.1 is an immediate consequence of Proposition 2.2 and part (b) and (c) of the preceding lemma. It remains to prove Corollary 1.3.

PROOF OF COROLLARY 1.3. Suppose that Γ is a distance-regular graph of diameter $d \geq 3$. Then q is an integer with $q \neq 0, -1$ (see Theorem 6.2.1 in [2]). Put $r := \begin{bmatrix} d \\ 1 \end{bmatrix}$.

First consider the case that $q \leq -2$. Then $\alpha < 0$, as $c_2 = (1+q)(1+\alpha)$. If d is even, then $r = (q^d - 1)/(q-1) < 0$. As $0 < b_0 = r\beta$ and $0 \leq \lambda = \beta - 1 + (r-1)\alpha$, it follows $1 - \alpha(r-1) \leq \beta < 0$. If d is odd, then $r > 0$ and hence $\beta > 0$. In this case $0 \leq a_2 = b_0 - b_2 - c_2 = (q+1)(\beta + r\alpha - 2\alpha - q\alpha - 1)$ gives $0 \leq \beta \leq 1 - \alpha(r - q - 2)$. Hence, for d even or odd and $q \leq -2$, we have $|\beta| \leq 1 + |\alpha|(|r| + |q| - 1)|$.

Now consider the case when $q \geq 1$ and $\alpha \neq q - 1, q$. Theorem 8.4.3 in [2] says that the multiplicity of the second largest eigenvalue of Γ is given by

$$f_1 = \frac{qr\beta\left(1 + \begin{bmatrix} d-2 \\ 1 \end{bmatrix}\alpha + q^{d-2}\beta\right)}{(\beta - \alpha + q)\left(1 + \alpha\begin{bmatrix} d-1 \\ 1 \end{bmatrix}\right)}.$$

As $c_2 = (1+q)(\alpha+1)$ and $a_1 = \beta - 1 + \alpha(r-1)$ are integers, we see that $\alpha' := \alpha(q+1)$ and $\beta' := \beta(q+1)$ are integers. This implies that $\beta' - \alpha' + q(q+1)$ is an integer that divides the integer $qr\beta'\big(q + 1 + \begin{bmatrix} d-2 \\ 1 \end{bmatrix}\alpha' + q^{d-2}\beta'\big)$. This implies that $\beta' - \alpha' + q(q+1)$ divides

$$\begin{aligned} qr(\alpha' - q(q+1))\left(q + 1 + \begin{bmatrix} d-2 \\ 1 \end{bmatrix}\alpha' + q^{d-2}(\alpha' - q(q+1))\right) \\ = (q+1)^2 r(r-1)(\alpha - q)(\alpha + 1 - q). \end{aligned}$$

If $\alpha \neq q - 1, q$, this yields the bound of Corollary 1.3, because $\beta' - \alpha' + q(q+1) = (q+1)(\beta - \alpha + q)$.

Next consider the case $q \geq 1$ and $\alpha = q - 1$. If $q = 1$, and $\beta > 3$, then it was shown by Egawa [5] that Γ is a Hamming graph. If $q \geq 2$, then part (a) Lemma 3.1 shows that Γ is a bilinear forms graph or that $\beta \leq (2q^4 + 2q^3 + 2q^2 + q - 1)r/(2q - 1)$.

Finally, consider the case that $q \geq 1$ and $\alpha = q$. If $q = 1$, then it was shown independently by Neumaier [10] and Terwilliger [11] that Γ is a Johnson graph. If $q \geq 2$ and $\beta > 8r(q^2 + 2q)/3$, then it follows easily from Theorem 2.3 in [9] that Γ is a Grassmann graph (in Theorem 2.3 of [9], define s to be the unique integer satisfying $4r/3 - 1 < s \leq 4r/3$). $\square$

REFERENCES

1. R. C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, *Pac. J. Math.*, **13** (1963), 389–419.
2. A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-regular Graphs*, Springer-Verlag, Berlin, Heidelberg, New York, 1989.
3. F. Buekenhout, Une charactérisation des éspaces affins basée sur la notation de droite, *Math. Z.*, **111** (1969), 367–371.
4. H. Cuypers, Two remarks on Huang's characterization of the bilinear forms graphs, *Europ. J. Combinatorics*, **13** (1992), 33–37.

5. Y. Egawa, Characterization of $H(n, q)$ by parameters, *J. Combin. Theory*, **A38** (1985), 1–14.
6. A. J. Hoffman, On eigenvalues and colourings of graphs, in: *Graph Theory and its applications*, B. Harris (ed.), pp. 79–91, Academic Press, New York, 1970.
7. T. Huang, A characterization of the association schemes of bilinear forms, *Europ. J. Combinatorics*, **8** (1987), 159–173.
8. K. Metsch, Improvement of Bruck's completion theorem, *Des. Codes and Cryptogr.*, **1** (1991), 99–116.
9. K. Metsch, A characterization of Grassmann graphs, *Europ. J. Combinatorics*, **16** (1995), 639–644.
10. A. Neumaier, A characterization of a class of distance-regular graphs. *J. Reine Angew. Math.*, **357** (1985), 182–192.
11. P. Terwilliger, The Johnson graph $J(d, r)$ is unique if $(d, r) \neq (2, 8)$, *Discrete Math.*, **58** (1986), 175–189.
12. J. A. Thas and F. de Clerck, Partial geometries satisfying the axiom of Pasch, *Simon Stevin*, **51** (1977), 123–137.

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